First Betti numbers of orbits of Morse functions on surfaces

Iryna Kuznietsova

(Institute of Mathematics of NAS of Ukraine, Tereshchenkivska str.,3, Kyiv, Ukraine) *E-mail:* kuznietsova@imath.kiev.ua

Yuliia Soroka

(Institute of Mathematics of NAS of Ukraine, Tereshchenkivska str.,3, Kyiv, Ukraine) *E-mail:* sorokayulya@imath.kiev.ua

Let \mathcal{G} be a minimal class of groups satisfying the following conditions: 1) $1 \in \mathcal{G}$; 2) if $A, B \in \mathcal{G}$, then $A \times B \in \mathcal{G}$; 3) if $A \in \mathcal{G}$ and $n \geq 1$, then the wreath product $A \wr_n \mathbb{Z} \in \mathcal{G}$.

In other words a group G belongs to the class \mathcal{G} iff G is obtained from trivial group by a finite number of operations \times , $\wr_n \mathbb{Z}$. It is easy to see that every group $G \in \mathcal{G}$ can be written as a word in the alphabet $\mathcal{A} = \{1, \mathbb{Z}, (,), \times, \wr_2, \wr_3, \wr_4, \ldots\}$. We will call such word a *presentation* of the group G in the alphabet \mathcal{A} . Evidently, the presentation of a group is not uniquely determined.

Denote by Z(G) and [G,G] the center and the commutator subgroup of G respectively.

Theorem 1. Let $G \in \mathcal{G}$, ω be an arbitrary presentation of G in the alphabet \mathcal{A} , and $\beta_1(\omega)$ be the number of symbols \mathbb{Z} in the presentation ω . Then there are the following isomorphisms:

$$Z(G) \cong G/[G,G] \cong \mathbb{Z}^{\beta_1(\omega)}.$$

In particular, the number $\beta_1(\omega)$ depends only on the group G.

The groups from the class \mathcal{G} appear as fundamental groups of orbits of Morse functions on surfaces. Let M be a compact surface and \mathcal{D} be the group of C^{∞} -diffeomorphisms of M. There is a natural right action of the group \mathcal{D} on the space of smooth functions $C^{\infty}(M, \mathbb{R})$ defined by the rule: $(f, h) \mapsto f \circ h$, where $h \in \mathcal{D}$, $f \in C^{\infty}(M, \mathbb{R})$. Let $\mathcal{O}(f) = \{f \circ h \mid h \in \mathcal{D}\}$ be the *orbit* of f under the above action. Endow the spaces \mathcal{D} , $C^{\infty}(M, \mathbb{R})$ with Whitney C^{∞} -topologies. Let $\mathcal{O}_f(f)$ denote the path component of f in $\mathcal{O}(f)$.

A map $f \in C^{\infty}(M, \mathbb{R})$ will be called *Morse* if all its critical points are non-degenerate. Homotopy types of stabilizers and orbits of Morse functions were calculated in a series of papers by Sergiy Maksymenko [3], [2], Bohdan Feshchenko [4], and Elena Kudryavtseva [1]. As a consequence of Theorem 1 we get the following.

Corollary 2. Let M be a connected compact oriented surface distinct from S^2 and T^2 , f be a Morse function on M, $G = \pi_1 \mathcal{O}_f(f) \in \mathcal{G}$, ω be an arbitrary presentation of G in the alphabet \mathcal{A} , and $\beta_1(\omega)$ be the number of symbols \mathbb{Z} in the presentation ω . Then the first integral homology group $H_1(\mathcal{O}(f),\mathbb{Z})$ of $\mathcal{O}(f)$ is a free abelian group of rank $\beta_1(\omega)$:

$$H_1(\mathcal{O}(f),\mathbb{Z})\simeq \mathbb{Z}^{\beta_1(\omega)}.$$

In particular, $\beta_1(\omega)$ is the first Betti number of $\mathcal{O}(f)$.

References

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